Robust Catalysis and Resource Broadcasting:
the possible and the impossible

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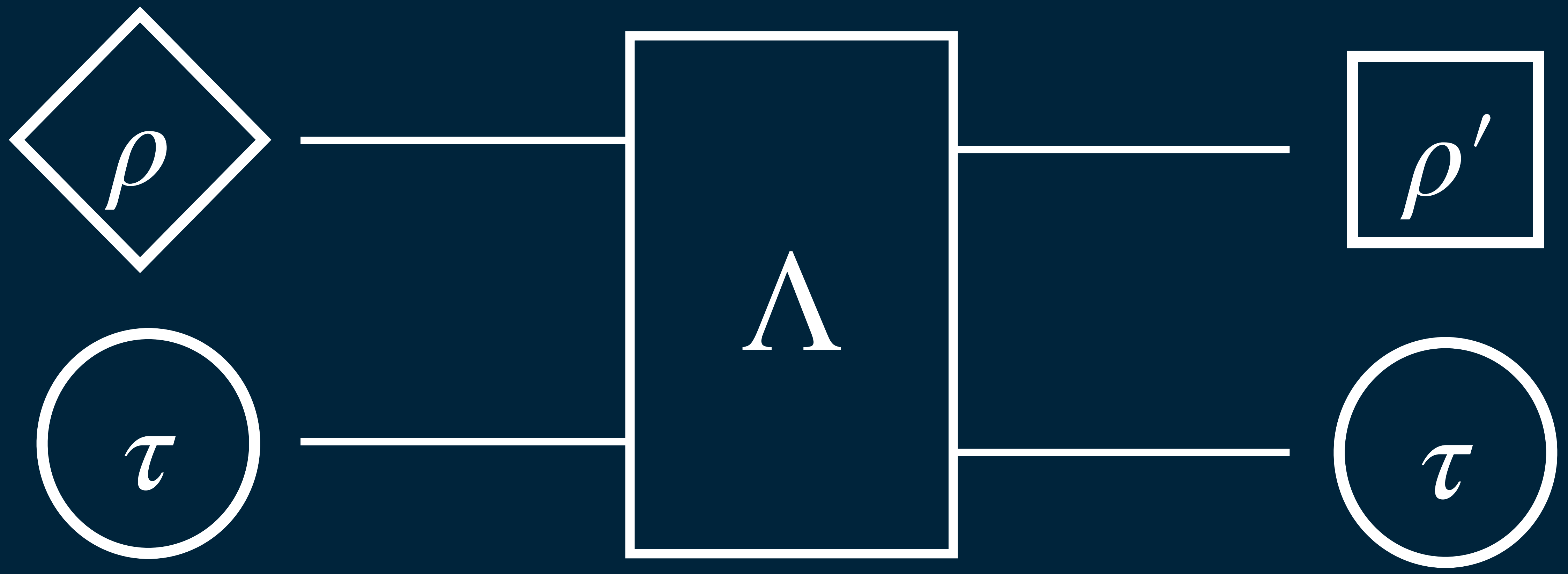
Catalysis in a nutshell

* various ways exist as to restricting the amount of error or correlation allowed in the final catalyst. In this work, we demand zero error on the final catalyst, and allow for system-catalyst correlations— but we will ask for more



An auxilliary system τ is used to facilitate a process, in a way that:

- the state transformation from $\rho \rightarrow \rho'$ would not have been possible otherwise (e.g. due to being constrained by a set of free operations $\mathcal{O}$)
- τ is returned* after the process, and therefore **recyclable**!

Thermodynamics

$\mathscr{O}$: Gibbs-state preserving operations *(or subsets thereof)*

$\mathscr{S}$: thermal Gibbs states

Generic quantum information

$\mathscr{O}$: set of all unital maps *(or subsets thereof)*

$\mathscr{S}$: maximally mixed states

Gaussianity

$\mathscr{O}$: Gaussian unitaries *(or subsets thereof)*

$\mathscr{S}$: Gaussian states

Others: Imaginarity, Complexity, etc

Compiling quantum circuits

$\mathscr{O}$: Clifford gates + measurement feed forward *(or subsets thereof)*

$\mathscr{S}$: stabiliser states

The essential resource-theoretic question

Given a fixed set of free channels $\mathscr{O}$, and free states $\mathscr{S}$ ($\mathscr{O}(\mathscr{S}) \subseteq \mathscr{S}$), when is $\rho \xrightarrow{\mathscr{O}} \rho'$ possible? E.g. entanglement.

The essential catalytic question

Given a fixed set of free channels $\mathscr{O}$, and free states $\mathscr{S}$, **and access to catalysts**, when is $\rho \xrightarrow{\mathscr{O}} \rho'$ possible? (State transition conditions will be relaxed)

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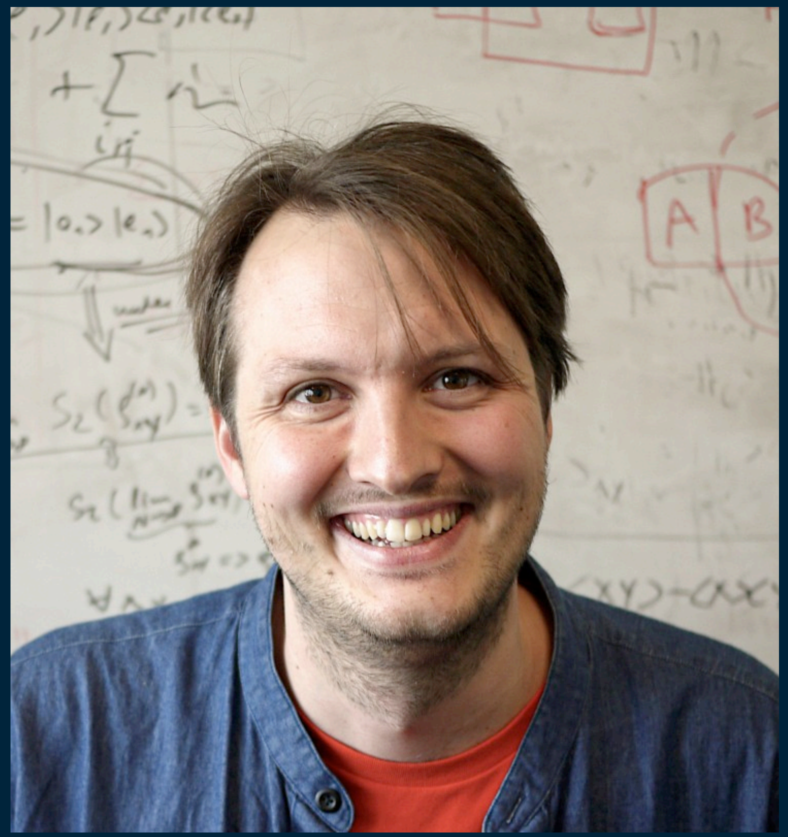

H. Wilming
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Featured in Physics

Catalysis in quantum information theory

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Catalysts open up new reaction pathways that can speed up chemical reactions while not consuming the catalyst. A similar phenomenon has been discovered in quantum information science, where physical transformations become possible by utilizing a quantum degree of freedom that returns to its initial state at the end of the process. In this review, a comprehensive overview of the concept of catalysis in quantum information science is presented and its applications in various physical contexts are discussed.

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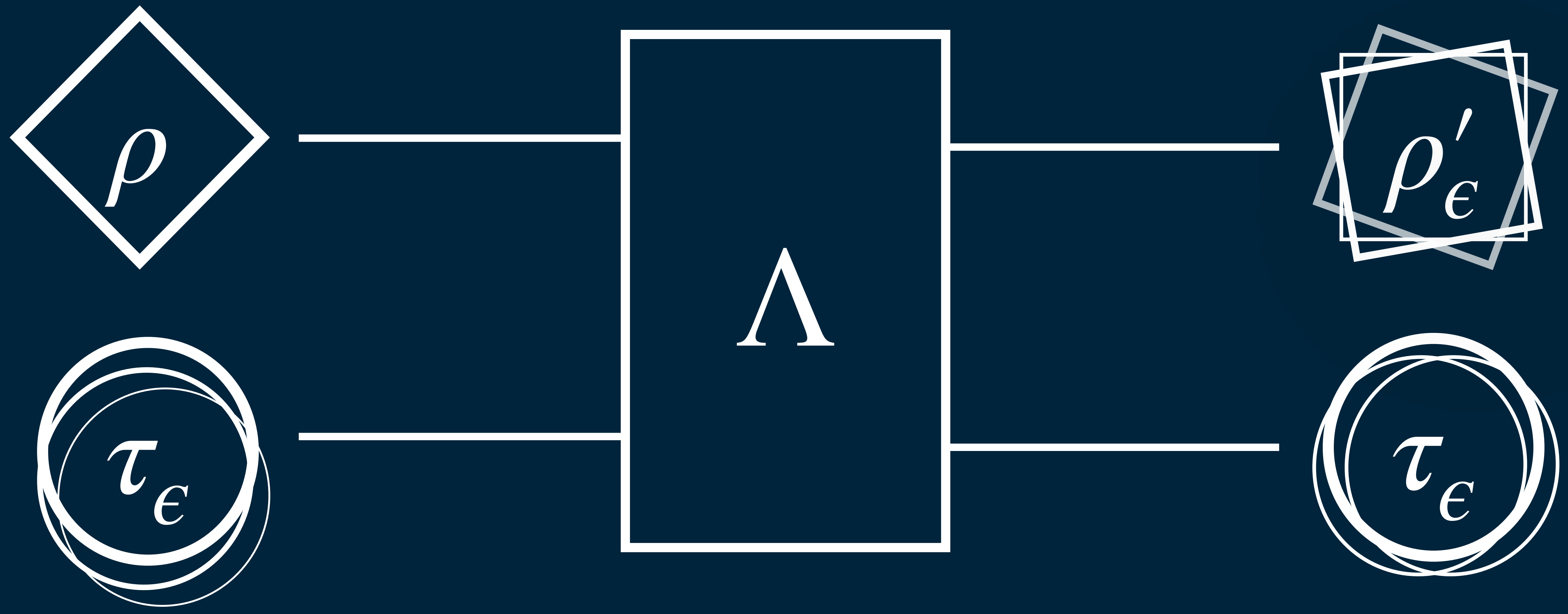
Catalytic processes are not quantum channels in general

- The catalytic condition that τ must be returned intact is usually demanded only when the system initial state is precisely ρ, and the channel describing their interaction is precisely Λ
- In other words, the catalytic condition is highly system-state (& interaction) dependent; the catalyst state is also heavily fine-tuned w.r.t. ρ.
  - On the one hand, this degree of freedom to choose the catalyst given ρ makes catalysis very powerful in relaxing state transition conditions.
  - On the other hand,
    - knowledge of system state can be costly
    - the catalyst state can be fragile w.r.t. errors

Catalysis in a fallen world

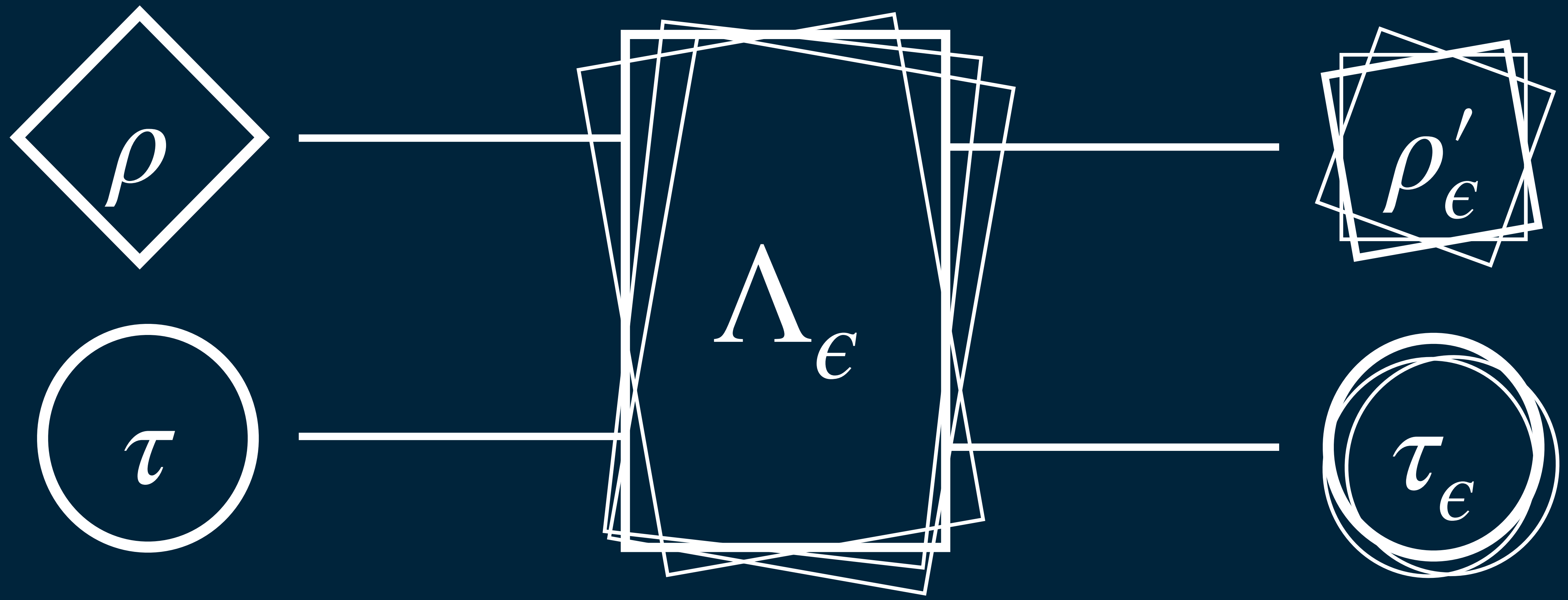
<u>Errors in preparation of the catalyst</u>



- final state transformation will be affected, but bounded by ϵ
- Data processing tells us that final catalyst error doesn't grow
- Reusing the catalyst in a second round is stable (because no fresh errors)

Catalysis in a fallen world

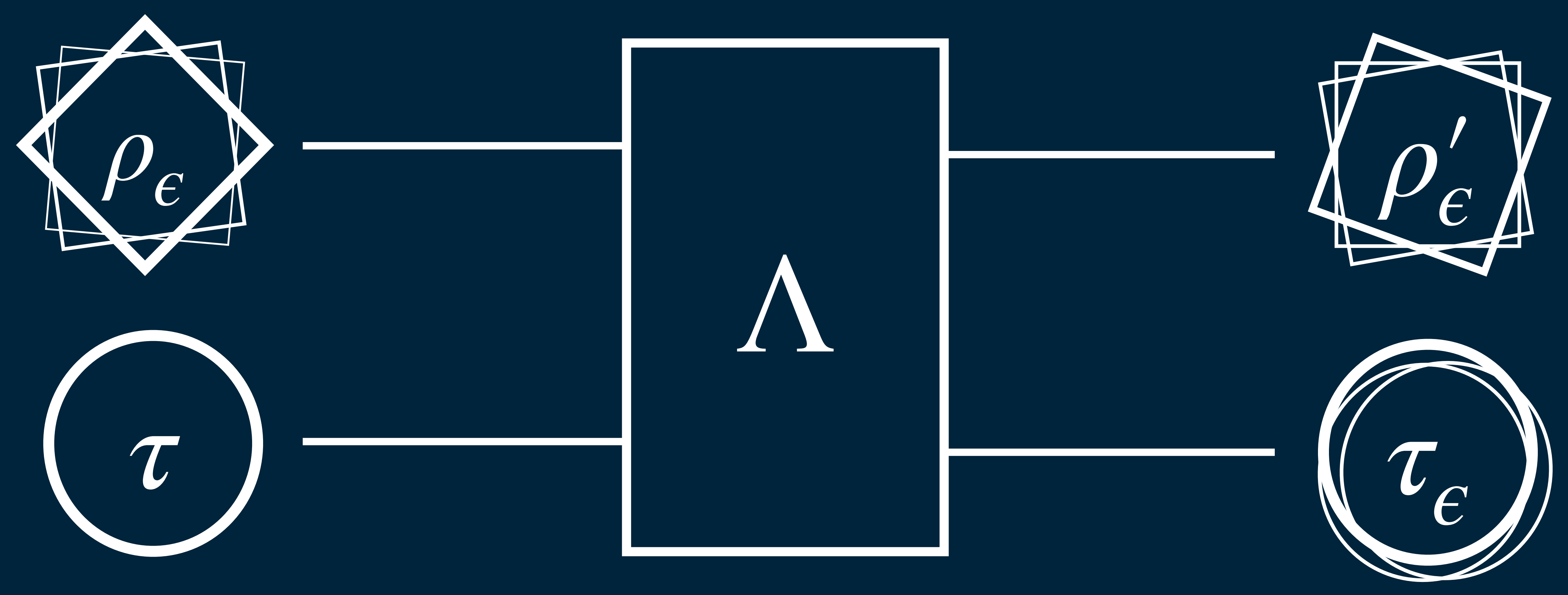
Errors in the process Λ



- Errors propagate into both system and catalyst
- Using the catalyst in a second round where fresh errors come in and accummulate
- No hope of controlling errors; we're out of luck :(

Catalysis in a fallen world

Errors in the initial state ρ



- fresh errors in the subsequent rounds can accumulate on the catalyst…..

unless the catalytic process Λ is already robust to such errors

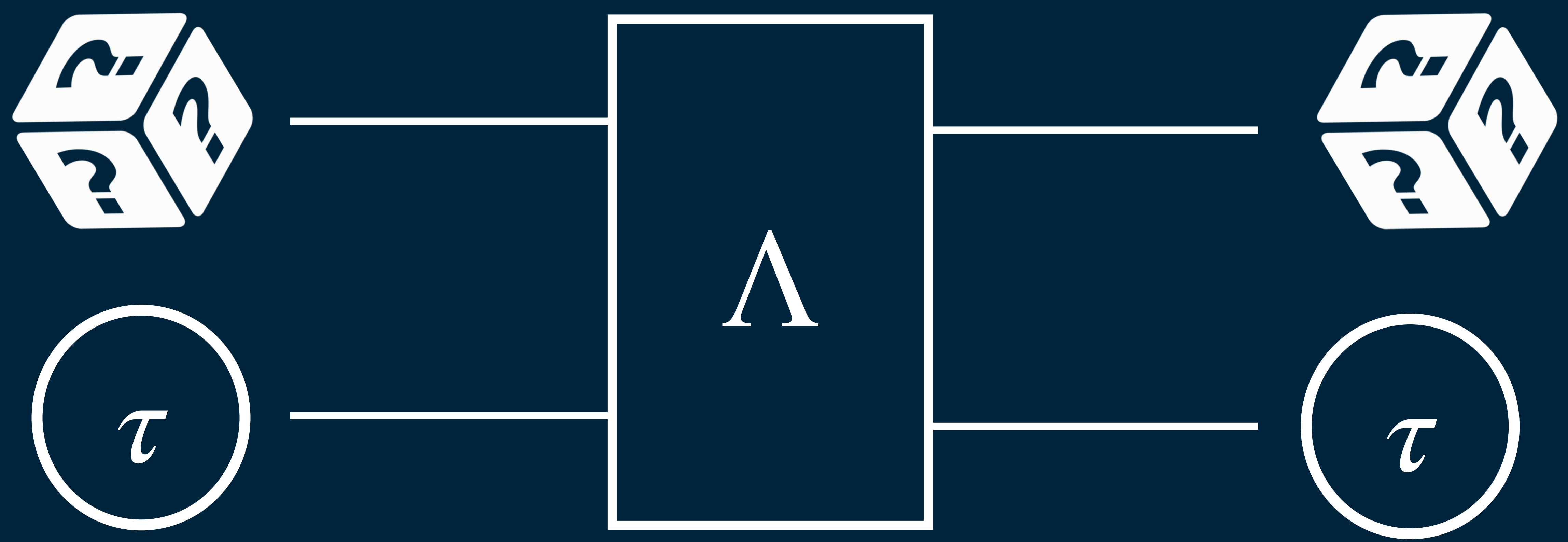
Catalytic quantum channels

A channel $\mathscr{E}_S$ is said to be a catalytic channel, if it has the form

$$\mathscr{E}_{S\to S'}(\,\cdot\,) = \mathrm{tr}_C\left[\Lambda_{SC\to S'C}(\,\cdot \otimes \tau_C)\right],$$

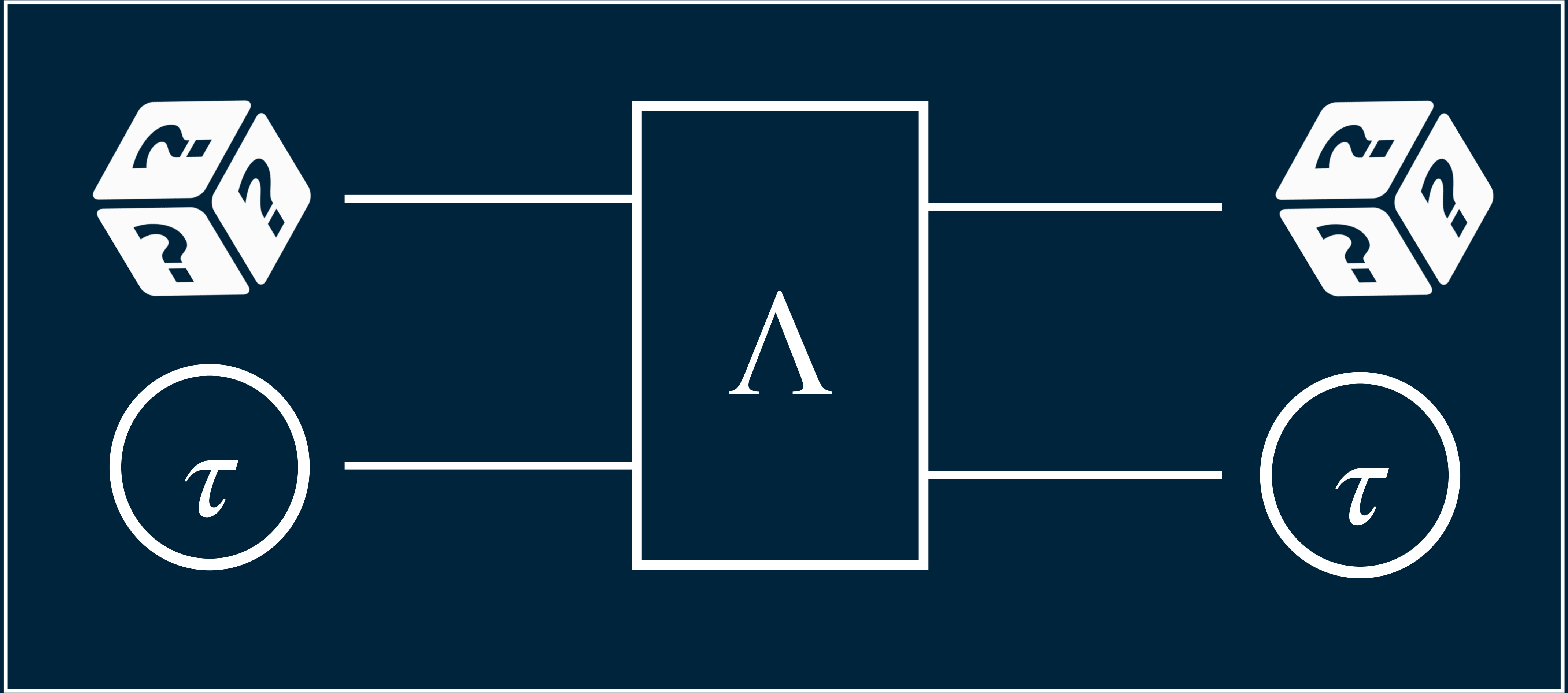
such that regardless of the input state ρ_S, we have that

$$\mathrm{tr}_{S'}\left[\Lambda_{SC\to S'C}(\,\cdot \otimes \tau_C)\right] = \tau_C.$$



Catalytic quantum channels

What else?



Ex 1: if $\mathscr{E}_S$ is a random unitary, then it can be implemented by an appropriate τ_C, and

$$U_{SC} = \sum_i U_S^{(i)} \otimes |i\rangle\langle i|_C$$

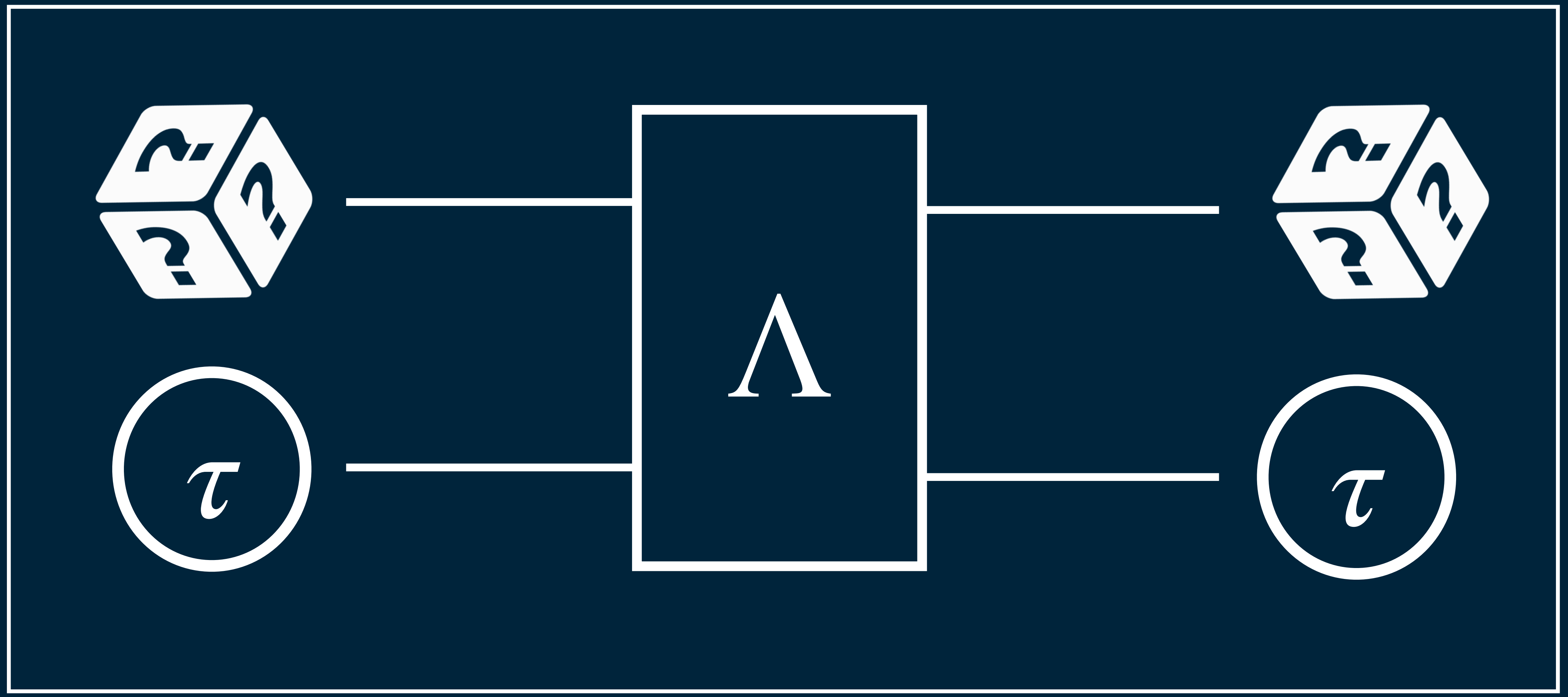
In literature one often considers the case where Λ is a unitary operation, e.g.

P. Boes, H.Wilming, R. Gallego, and J. Eisert, PRX 8, 041016 (2018)
S. H. Lie and H. Jeong, PRR 3, 013218 (2021).

Here, we defined it w.r.t. a channel chosen from a set of free operations $\mathcal{O}$

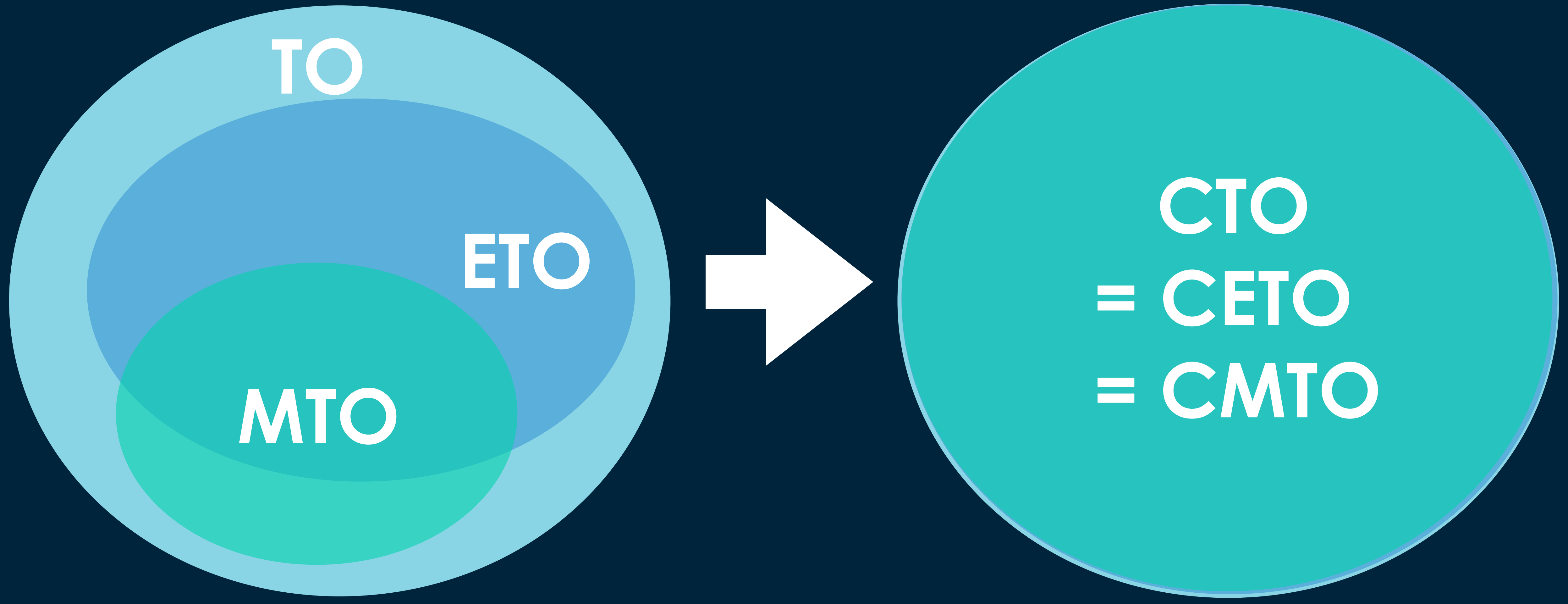
Catalytic quantum channels

Hmm... is it not too restrictive still?



Here, we defined it w.r.t. a channel chosen from a set of free operations $\mathcal{O}$

Ex 2: if a catalyst state τ can be prepared via free operations on C, then all $\Lambda_{SC\to S'C}$ can be made catalytic by concatenating with this preparation channel

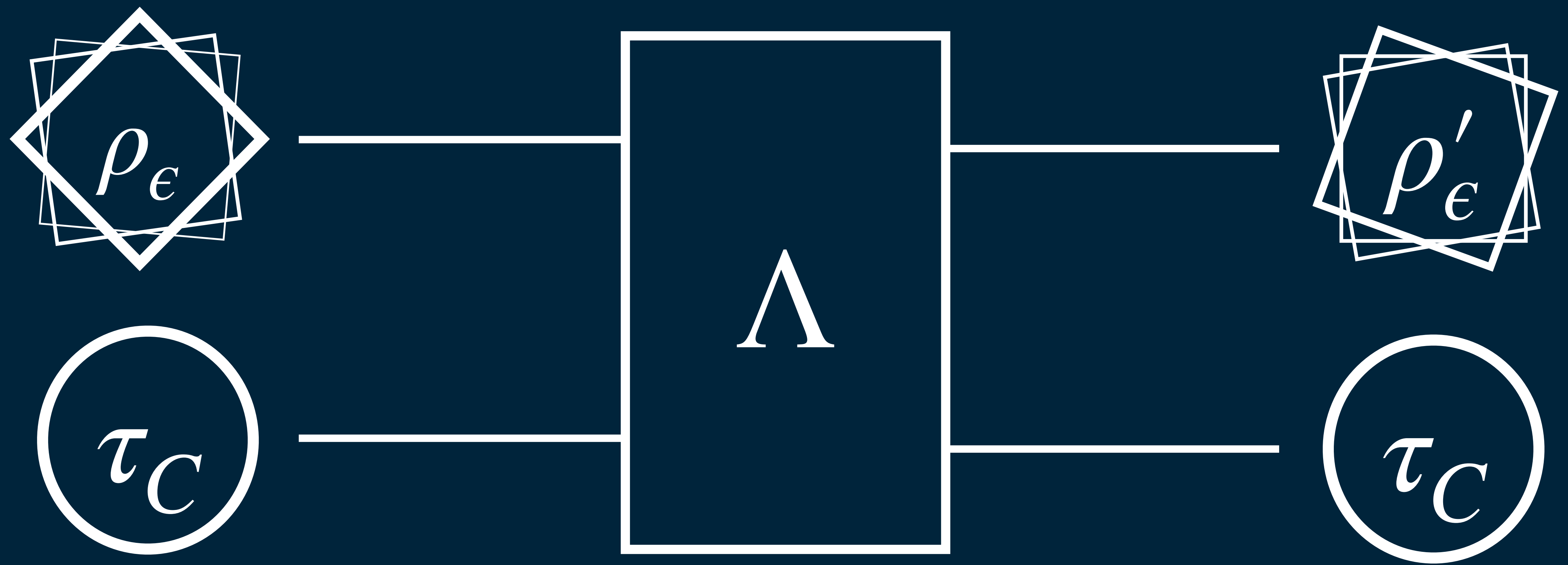


Gibbs states can be catalysts

K Korzekwa, M Lostaglio
PRL 129 (4), 040602, 2022

Jeongrak Son, Nelly Ng,
Quantum Science and Technology 10 (1), 015011, 2024

Robust catalysis



Def: A channel $\Lambda_{SC\to S'C}$ implements (ρ, ϵ)-robust catalysis if there exists a catalyst τ, such that $\mathrm{Tr}_{S'}[\Lambda(\sigma_S \otimes \tau_C)] = \tau_C$ **for all σ_S that are ϵ-close to ρ_S (e.g. in trace distance)**

- system-catalyst correlation is allowed
- exact recovery of τ_C
- does not care about the system final state

$\epsilon = 0$: Fragile, fine-tuned catalysis

ϵ-robust catalysis

$\epsilon = 1$: Catalytic channel

ϵ-robustness seems physically motivated and perhaps it will be a good way to interpolate between a highly fragile, fine-tuned catalysis, versus a highly restrictive catalytic channel?

Unfortunately, no… this is **Result 1**:
ϵ-robustness for any $\epsilon > 0$ is equivalent to demanding a catalytic channel

Robust catalysis = catalytic channels !

Bad news :(

Any degree of robustness as governed by the tolerable error is already equivalent to maximal robustness

Good news :)

Physical motivation of robust catalysis is quite strong; and turns out that it can be studied purely on the level of the channels (rather than state transformations)

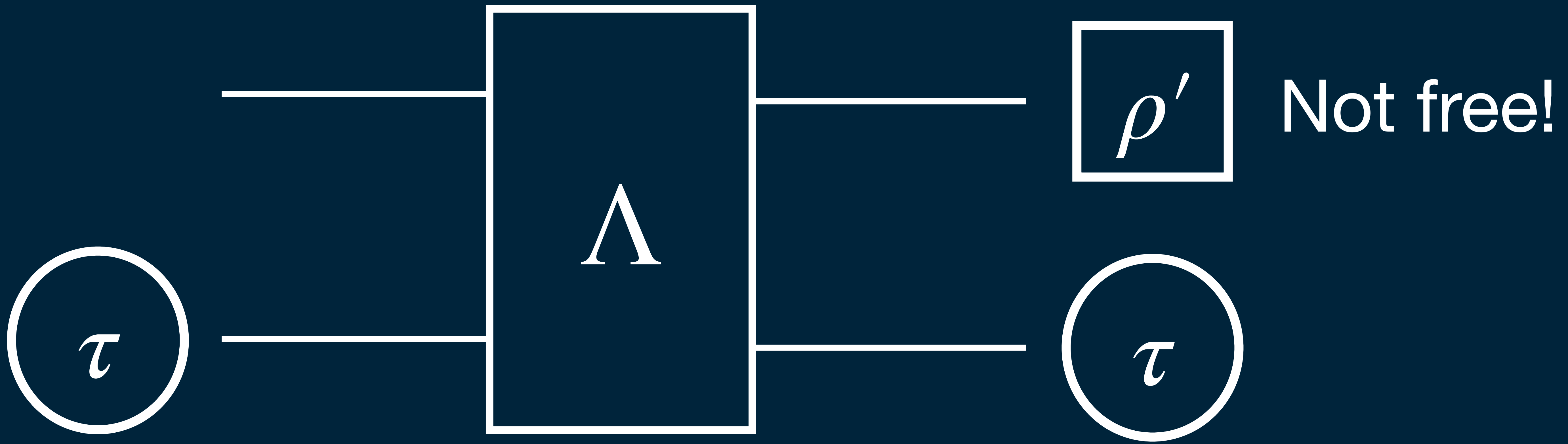
There is a clearer reason now to elucidate properly what can and cannot be achieved by catalytic channels

When does robust catalysis provide advantage?

- In other words, when does a free operation Λ_{SC}, when coupled with a catalyst τ_C used robustly, induces a channel $\Lambda_S(\,\cdot\,) = \mathrm{Tr}_C[\Lambda_{SC}(\,\cdot \otimes \tau_C)]$ that is not free?
- We wanted to study this problem beyond a particular type of resource theory…
- Approach: start from a set of free states $\mathcal{S}$, and consider the set of completely resource non-generating (CRNG) operations to be free

RNG operations:	$\mathcal{O}(\mathcal{S}) \in \mathcal{S}$	Separable operations, Gibbs-preserving operations, covariant operations etc
CRNG operations:	$\mathcal{O} \otimes \mathbf{I}$ is RNG	

Broadcasting in a nutshell



Note a striking resemblence with robust catalysis!

Robust Catalysis exists

Resource Broadcasting exists

The usage of auxilliary system τ to prepare a non-free state ρ' from scratch, where τ is returned exactly after the process

Subtle difference: conventionally, the input is often assumed to be a fixed (but free) state $\rho \in \mathcal{S}$. The broadcasting process might fail to be robust catalytic if this is not the case.

Robust catalysis vs broadcasting

Technically, these processes have been motivated and defined differently, and their differences are subtle. Nevertheless, their conceptual link can be strengthened:

Result 2

Given a CRNG resource theory satisfying some basic axioms (e.g. tensor product and marginals of free states are also free, $\mathcal{S}$ is convex)

Robust Catalysis exists $\Longleftrightarrow$ Resource Broadcasting exists

Simplifies the analysis for RC (at least for CRNGs)!

When is robust catalysis/broadcasting (im)possible in CRNG theories?

Perhaps the properties of generic resource monotones can tell us something useful……

Recall: a function $R(\rho)$ is a resource monotone only if for any state ρ and any free operation $\Lambda \in \mathcal{O}$, $R(\rho) \geq R(\Lambda(\rho))$.

In particular, a very useful and common property of resource monotone is super-additivity, i.e.

$R(\rho_{AB}) \geq R(\rho_A) + R(\rho_B)$ for all ρ_{AB}.

Super-additive and tensor-product additive monotones remain monotones under correlated catalysis
N. Shiraishi, R. Takagi
Phys. Rev. Lett. 132, 180202 (2024)

What does super-additivity tell us?

Suppose we find the existence of a **super-additive and faithful** resource monotone. This means

→ No Resource Broadcasting

$R(\rho_{AB}) \geq R(\mathrm{Tr}_B[\rho_{AB}]) + R(\mathrm{Tr}_A[\rho_{AB}])$ for all states ρ_{AB}.

Suppose we start from an initial state $\gamma_S \otimes \tau_C$, where γ_S is a free state. Suppose that there exists a free operation $\Lambda_{SC \to S'C}$ such that $\Lambda(\gamma_S \otimes \tau_C) = \chi_{S'C}$.

1) R is faithful, hence $R(\gamma_S) = 0$.
2) R is a super-additive monotone, hence
   $R(\tau_C) = R(\gamma_S \otimes \tau_C) \geq R(\chi_{S'C}) \geq R(\chi_{S'}) + R(\tau_C)$.

1) + 2) imply that $R(\chi_{S'}) \leq 0$ which by faithfulness implies $\chi_{S'}$ is free

Existence of a faithful, super-additive resource monotone $\Rightarrow$ No Resource Broadcasting $\overset{\text{CRNG}}{\Longleftrightarrow}$ No Robust Catalysis!

Known examples where the resource theory has a faithful, super-additive resource monotone: athermality, coherence, entanglement, PPT entanglement, magic, optical nonclassicality

The converse is not true: e.g. the theory of asymmetry (connected Lie groups) where there is no such resource monotone, but no-broadcasting holds

I. Marvian and R. W. Spekkens, PRL 123, 020404 (2019)
M. Lostaglio and M. Müller, PRL 123, 020403 (2019).

Result 3 : the impossible — a generic CRNG condition that gives no-broadcasting

This relates to how the set of free states are extended when we compose different quantum systems.

Suppose a CRNG resource theory has the following composition rule on the set of free states: given individual sets of free states $\mathcal{S}_A$, $\mathcal{S}_B$, the set of composite free states are given by

$$\mathcal{S}_{AB} = \mathcal{S}_A \otimes_{\min} \mathcal{S}_B := \{\mathrm{conv}(\rho_A \otimes \rho_B) \,|\, \rho_A \in \mathcal{S}_A, \rho_B \in \mathcal{S}_B\},$$

then **neither resource broadcasting nor robust catalysis is allowed**.

- Proof is based on earlier intuition of super-additive monotones
- Composition rules deserve a better discussion which we skip for now

The possible and the impossible

	Robust Catalysis and Resource Broadcasting	
CRNG resource theories	Is this ever possible?	**No** Athermality ($T > 0$) [Thm. 3] [56] MIO Coherence [Thm. 3] [57] Entanglement [59, 60] PPT entanglement [61] Magic [20] Asymmetry (connected Lie groups) [17, 18] Optical nonclassicality [62]
	Robust Catalysis	
non-CRNG resource theories	Is this ever possible?	

Is robust catalysis/broadcasting ever possible?

Broadcasting of imaginarity known

R. Takagi, T. J. Yoder, and I. L. Chuang, PRA 96, 042302 (2017).

L. Zhang and N. Li, Commun. Theor. Phys. 76, 115104 (2024).

Catalytic replication

K. Kuroiwa, H. Yamasaki, Quantum 4, 355 (2020)

$$\tau \longrightarrow \tau\ \tau \longrightarrow \tau\ \tau\ \tau \longrightarrow \ldots\ldots$$

Happens when there is no full rank free state — we show in general that this always gives robust catalytic advantage

Result 4 : the possible — generic conditions on CRNGs that result in broadcasting

Suppose a CRNG resource theory has the following composition rule on the set of free states: given individual sets of free states $\mathcal{S}_A, \mathcal{S}_B$,

the set of composite free states are given by

$$\mathcal{S}_{AB} = \mathcal{S}_A \otimes_{\max} \mathcal{S}_B := \{\rho_{AB} \,|\, \rho_A \in \mathcal{S}_A, \rho_B \in \mathcal{S}_B\},$$

or

$$\mathcal{S}_{AB} = \mathcal{S}_A \otimes_{\mathrm{sep}} \mathcal{S}_B := \mathcal{S}_A \otimes_{\max} \mathcal{S}_B \cap \mathrm{SEP}_{\mathrm{AB}}$$

then it's easy to get broadcasting.

Result 4 : the possible — generic CRNGs with broadcasting (cont.)

For any theory satisfying the composition rule of $\mathcal{S}_A \otimes_{\mathrm{sep}} \mathcal{S}_B$ or $\mathcal{S}_A \otimes_{\max} \mathcal{S}_B$, if on the catalyst, we have a single free state $\mathcal{S}_C = \{\gamma_C\}$, and let $r = \max_{\tau_C} D_{\max}(\tau_C \| \gamma_C)$. Then,

σ_S is a state such that $D_{\max}(\sigma_S \| \mathcal{S}_S) \leq r$

$\Updownarrow$

σ_S can be prepared by a broadcasting map

The possible and the impossible

	Robust Catalysis and Resource Broadcasting	
	Yes	**No**
CRNG resource theories	Athermality ($T = 0$) [48] Imaginarity [45] Asymmetry (finite groups) [58] Thm. 4 Limited subspace theories [Supplemental Materials Sec. IV D]	Athermality ($T > 0$) [Thm. 3] [56] MIO Coherence [Thm. 3] [57] Entanglement [59, 60] PPT entanglement [61] Magic [20] Asymmetry (connected Lie groups) [17, 18] Optical nonclassicality [62]
	Robust Catalysis	
	Yes	**No**
non-CRNG resource theories	Elementary thermal operations [34, 35] Markovian thermal operations [32, 33, 35] Unitary operations [13–15]	Gibbs preserving covariant operations ($T > 0$) [Supplemental Materials Sec. IV B] Thermal operations ($T > 0$) [63]

Current status quo

Take-home message

- There is good reason to study catalysis under the robustness condition, namely, to avoid the fragility of catalytic recovery conditions — this is the right way to deal with errors.

- More robust catalytic advantage for non-CRNG free operations?
    - Note: if CRNG is no-broadcasting, then all subsets are no-broadcasting. But this is not true for RC :) we have two main examples, but is there more?

Thanks for listening!